The substitution method

Recall that, if f and g are differentiable functions, then by the chain rule

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$$

Thus we have $\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C$

If we let $u = g(x)$, then we have that

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C = f(u) + C = \int f'(u)\,du$$

This suggests that given an indefinite integral of the form

$\int f'(g(x))g'(x)\,dx$, we can turn this integral into $\int f'(u)\,du$ by letting $u = g(x)$ and it may be that the latter integral is easier to find.

Example: Find $\displaystyle\int \frac{e^x + \sin x}{e^x - \cos x}\,dx$

Solution:

$$\int \frac{e^x + \sin x}{e^x - \cos x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|e^x - \cos x| + C$$

$$\boxed{\begin{array}{l} e^x - \cos x = u \\ (e^x + \sin x)dx = du \end{array}}$$

- $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{(-1)\cdot du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

$$\boxed{\begin{array}{l} \cos x = u \\ -\sin x\,dx = du \end{array}}$$

- $$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

$$\boxed{\begin{array}{l} \sin x = u \\ \cos x\,dx = du \end{array}}$$

- $\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}\,dx = \int \frac{\sec^2 x+\sec x\tan x}{\tan x+\sec x}\,dx = \int \frac{du}{u} = \ln|u|+C$
  $= \ln|\sec x+\tan x|+C$

  $\boxed{\begin{array}{l}\tan x+\sec x = u\\ (\sec^2 x+\sec x\tan x)dx = du\end{array}}$

- $\int \csc x\,dx = -\ln|\csc x+\cot x|+C$ apply the same trick with $(\csc x+\cot x)$

Theorem: let g be differentiable on (a,b) and continuous on $[a,b]$ and suppose that f is continuous on the range of g. Then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$u=g(x)$
$du = g'(x)\,dx$

Proof: let $F(x)$ be an antiderivative for $f(x)$. Then $\big(F(g(x))\big)' = F'(g(x))\cdot g'(x)$
$= f(g(x))\cdot g'(x)$

and so, by FTC, $\int_a^b f(g(x))\cdot g'(x)\,dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$

$= F(u)\Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,du$ ▨

Example: Find $\int_0^3 (x+1)\sqrt{x+2}\,dx$

Solution: $\int_0^3 (x+1)\sqrt{x+2}\,dx = \int_2^5 (u-1)\sqrt{u}\,du = \int_2^5 (u\sqrt{u}-u)\,du$

$\boxed{\begin{array}{l}x+2=u\\ 1.\,dx = du\end{array}}$ $\boxed{\begin{array}{l}0+2=2\\ 3+2=5\end{array}}$

$= \int_2^5 u^{\frac{3}{2}} - u^1\,du = \left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^2}{2}\right)\Bigg|_2^5$

$= \left(\frac{2}{5}5^{\frac{5}{5}} - \frac{25}{2}\right) - \left(\frac{2}{5}2^{\frac{5}{2}} - 2\right)$

Example: Find $\int_1^7 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution: $\displaystyle\int_1^7 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_{x=1}^{x=7} \frac{2}{\ln 2}\,du = \left(\frac{2}{\ln 2}u\right)\Bigg|_{x=1}^{x=7} = \left(\frac{2}{\ln 2}2^{\sqrt{x}}\right)\Bigg|_{x=1}^{x=7}$

$$\boxed{\begin{gathered} 2^{\sqrt{x}} = u \\ \ln 2 \cdot 2^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx = du \end{gathered}}$$

$$= \frac{2}{\ln 2}2^{\sqrt{7}} - \frac{2}{\ln 2}2$$

Some trigonometric integrals

Example: Find $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^5 x\,dx$.

Solution: $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^5 x\,dx = \int_0^{\pi/2} \sin^7 x\, \underbrace{\cos^4 x}_{(1-\sin^2 x)^2}\, \cos x\,dx = \int_0^1 u^7 \cdot (1-u^2)^2\,du = \int_0^1 u^7(1-2u^2+u^4)\,du$

$$\boxed{\begin{gathered} \sin x = u \\ \cos x\,dx = du \end{gathered}}$$

$$= \int_0^1 u^7 - 2u^9 + u^{11}\,du$$

$$= \left(\frac{u^8}{8} - \frac{2u^{10}}{10} + \frac{u^{12}}{12}\right)\Bigg|_0^1$$

$$= \frac{1}{8} - \frac{2}{10} + \frac{1}{12}$$

$$\| $$

$$\int_0^{\pi/2} \sin^6 x \cos^5 x \sin x\,dx \quad (\sin^6 x = (1-\cos^2 x)^3)$$

$$\|$$

$$\int_1^0 (1-u^2)^3 u^5 (-1)\,du \qquad \boxed{\begin{gathered} \cos x = u \\ -\sin x\,dx = du \end{gathered}}$$

$$\|$$

$$\vdots$$

Exercise!

Example: Find $\displaystyle\int \cos^4 x\,dx$

Solution: $\displaystyle\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$\boxed{\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \quad \Longrightarrow \quad \cos^2 x = \frac{1+\cos 2x}{2} \\ &= 1 - 2\sin^2 x \qquad\qquad \sin^2 x = \frac{1-\cos 2x}{2} \end{aligned}}$$

$$= \int \frac{1 + 2\cos 2x + \cos^2 2x}{4}\,dx = \int \frac{1}{4}\,dx + \int \frac{1}{2}\cos 2x\,dx + \int \frac{\cos^2 2x}{4}\,dx$$

$$\boxed{\begin{array}{l} 2x = u \\ 2dx = du \end{array}}$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \int \frac{1}{2}\,\frac{\cos^2 u}{4}\,du$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8}\int \frac{1 + \cos 2u}{2}\,du$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8}\left(\frac{u}{2} + \frac{\sin 2u}{4}\right) + C$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8}\left(x + \frac{\sin 4x}{4}\right) + C$$

Example: Find $\int_0^{\pi/3} \sin^4 x \cos^2 x\,dx$

Solution: $\int_0^{\pi/3} \sin^4 x \cos^2 x\,dx = \int_0^{\pi/3} \left(\frac{1 - \cos 2x}{2}\right)^2 \left(\frac{1 + \cos 2x}{2}\right) dx$

$$= \int_0^{\pi/3} \left(\frac{1 - 2\cos 2x + \cos^2 2x}{4}\right)\left(\frac{1 + \cos 2x}{2}\right) dx$$

$$= \int_0^{\pi/3} \frac{1}{8}\left(1 + \cancel{\cos 2x} - \cancel{2}\cos 2x - 2\cos^2 \cancel{2x} + \cancel{\cos^2} 2x + \cos^3 2x\right) dx$$

$$-\cos 2x \qquad -\cos^2 2x$$

$$= \int_0^{\pi/3} \frac{1}{8}\left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$\boxed{\begin{array}{l} u = 2x \\ du = 2dx \end{array}}$$

Exercise!

$$= \int_0^{2\pi/3} \frac{1}{8}\left(1 - \cos u - \cos^2 u + \cos^3 u\right) du = \dots$$

Example: Find $\int \sec^3 x \tan^5 x \, dx$

Exercise!

Solution: $\int \sec^3 x \tan^5 x \, dx = \int \sec^2 x \tan^4 x \, \sec x \tan x \, dx = \int u^2 \cdot (u^2-1)^2 \, du = \text{---}$

$$\boxed{\begin{aligned} \sec x &= u \\ \sec x \tan x \, dx &= du \end{aligned}}$$

$$\boxed{\begin{aligned} 1+\tan^2 x &= \sec^2 x \\ \tan^2 x &= u^2-1 \end{aligned}}$$